UNIT- 3

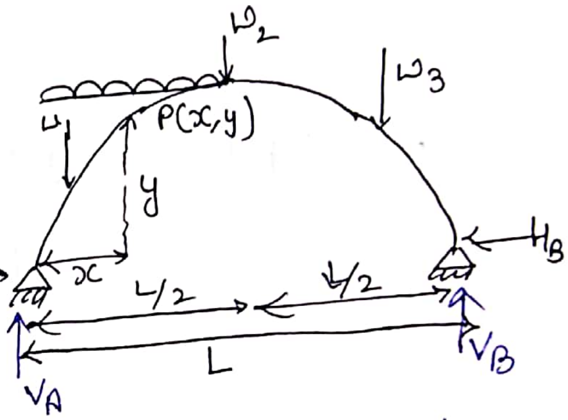
TWO HINGED ARC

Two Hinge Arches :

- A two hinge arch is a indeterminate structure having degree of indeterminacy 1.
- The two hinge arch is mainly of two types :
  i) Circular two hinged arch.
  ii) Parabolic two hinged arch.

Analysis of two hinge arches :

- A two hinge arch is shown in the figure, there are 4 reaction components ie. V_A, V_B, H_A & H_B.



- Normally arches are subjected to vertical forces only. Hence the horizontal forces are taken as equillibrium (i.e $H_A = H_B = H$).
- The expression for horizontal force can be found out by using 2 methods :
  1) By using castigliano 1^{st} theorem
  2) By Unit Load method.

- **Castigliano's 1st theorem:**

- Consider an arch as shown above in figure and Let M_x be the bending moment at point P on arch from support A.

- Let M be the moment due to all vertical loads at P & H is horizontal thrust.

- So Mathematically:

$$M_x = M' - Hy$$

- So, acc. to castigliano's first theorem

$$\frac{\partial U}{\partial H} = \text{Horizontal displacement}$$

$$= 0 \text{ (since supports are unyielding)}$$

- Expression for strain energy will be

$$U = \int \frac{M_x^2 ds}{2EI}$$

$$= \int \frac{(M' - Hy)^2 ds}{2EI}$$

- Put the value of strain strain energy in theorem:

$$\frac{\partial}{\partial H} \int \frac{(M' - Hy)^2 ds}{2EI} = 0$$

$$\int 2(M' - Hy)(-y) \frac{ds}{2EI} = 0$$

$$\int (-M'y + Hy^2) \frac{ds}{EI} = 0$$

$$H \int y^2 \left(\frac{ds}{EI}\right) = \int M'y \left(\frac{ds}{EI}\right)$$

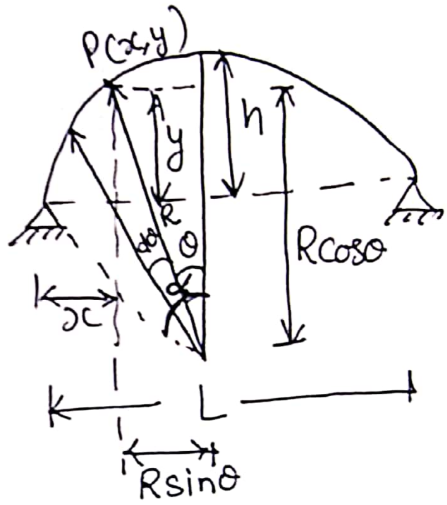
$$\boxed{H = \frac{\int M' y \frac{ds}{EI}}{\int y^2 \frac{ds}{EI}}}$$

- **Analysis of two hinge arch (circular):**

- $ds = R\,d\theta$

$y = h - (R - R\cos\theta)$

$$\boxed{y = h - R(1 - \cos\theta)}$$



- $$\boxed{x = \frac{L}{2} - R\sin\theta}$$

- $$\boxed{\frac{L}{2} \times \frac{L}{2} = h(2R - h)}$$

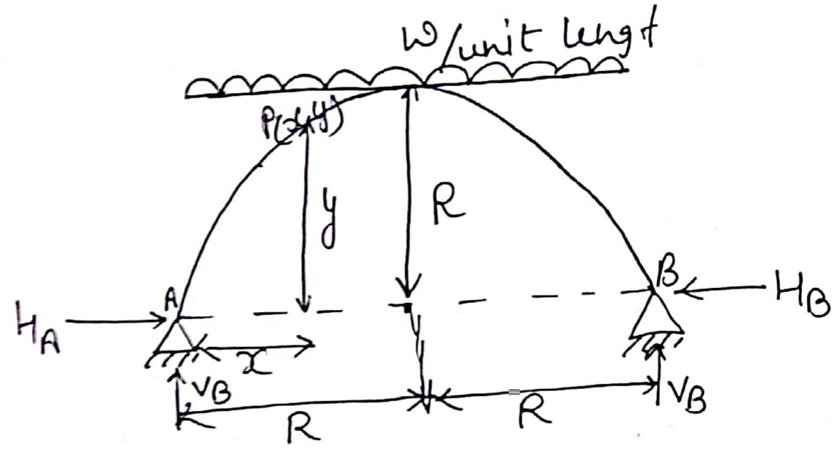
- **Normal thrust (N)**

$$\boxed{N = V\sin\theta + H\cos\theta}$$

- **Radial Shear (Q)**

$$\boxed{Q = V\cos\theta - H\sin\theta}$$

Q-1 A semi circular arch of radius R is subjected to UDL of w/unit length over entire span. Assuming EI to be constant Determine the horizontal thrust.



Sol :- (1) → $V_A = V_B = wR$ $\boxed{ds = R\,d\theta}$

(2) → $y = h - (R - R\cos\theta)$ $\boxed{h = R}$

$= R - (R - R\cos)$

$y = R\cos\theta$

(3) $x = R - R\sin\theta$

$= R(1 - \sin\theta)$

(4) $M' = V_A x - \frac{wx \times x}{2}$

$= wR(R - R\sin\theta) - \frac{w}{2}\left[R - R\sin\theta\right]^2$

$= wR^2\left[(1-\sin\theta) - \left(\frac{1-\sin\theta)^2}{2}\right)\right]$

$= \frac{wR^2}{2}(1-\sin\theta)\left[2 - (1-\sin\theta)\right]$

$= \frac{wR^2}{2}(1+\sin\theta)(1-\sin\theta)$

$\frac{wR^2}{2}(1-\sin^2\theta) = \frac{wR^2}{2}\cos^2\theta$

⑤ $\int M' y\, ds = \int_{-\pi/2}^{\pi/2} \frac{WR^2}{2} \cos^2\theta \, R\cos\theta \, R\, d\theta$

$= \int_{-\pi/2}^{\pi/2} \frac{WR^4}{2} \cos^3\theta \, d\theta$

$= \frac{WR^4}{2} \int_{-\pi/2}^{\pi/2} (1-\sin^2\theta)\cos\theta \, d\theta$

$=$ Put $t = \sin\theta$ | $t = -1$ to 1

$dt = \cos\theta \, d\theta$

$= \frac{WR^4}{2} \int_{-1}^{1} (1-t^2)\, dt$

$= \frac{WR^4}{2} \left[t - \frac{t^3}{t} \right]_{-1}^{1}$

$= \frac{WR^4}{2} \left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right]$

$= \frac{WR^4}{2} \left(\frac{4}{3}\right) = \frac{2WR^4}{3}$

6) $\int y^2 ds = \int_{-\pi/2}^{\pi/2} R^2 \cos^2\theta \, R\, d\theta$

$= \int_{-\pi/2}^{\pi/2} R^3 \left(\frac{1+\cos 2\theta}{2} \right) d\theta$

$= \frac{R^3}{2} \left[\theta + \sin 2\theta \right]_{-\pi/2}^{\pi/2}$

$= \frac{R^3}{2} \left[\left(\frac{\pi}{2} + 0\right) - \left(-\frac{\pi}{2} - 0\right) \right]$

$= \frac{R^3 \pi}{2}$

7) $\boxed{H = \frac{\int M' y\, ds}{\int y^2 ds} = \frac{2WR^{\cancel{4}}}{3} \times \frac{2}{\cancel{R^3}\pi} = \frac{4WR}{3\pi}}$

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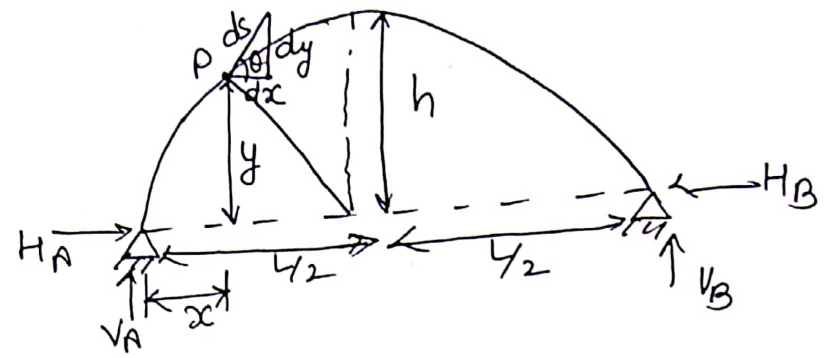
Two hinge Parabolic Arc :-



- $y = \dfrac{4hx(L-x)}{L^2}$

Let $P(x,y)$ be a point on the arch and ds be the elemental length.

Let θ be the slope of arch to the horizontal at P then

$$\boxed{\tan\theta = \frac{dy}{dx}} \qquad \boxed{\cos\theta = \frac{dx}{ds}} \qquad \boxed{ds = dx\sec\theta}$$

- Horizontal thrust

$$\boxed{H = \frac{\int M' y\, ds}{\int y^2\, ds}}$$

- so parabolic arc $\quad ds = dx \sec\theta$

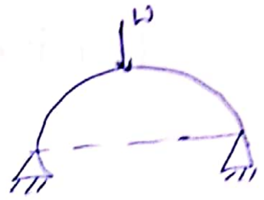
$$H = \frac{\int M' y\, dx \sec\theta}{\int y^2\, dx \sec\theta}$$

$$\boxed{H = \frac{\int M' y\, dx}{\int y^2\, dx}}$$

• Conditions For Horizontal thrust (H) =

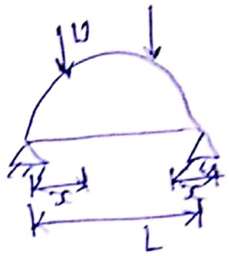
a) If a 2 hinge parabolic arch of span L & Rise h carrying a point load w at the crown the

$$\boxed{H = \frac{25}{128}\left[\frac{wL}{h}\right]}$$



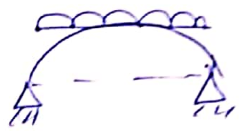
b) If Load is at a distance x from a springing (end point) then

$$\boxed{H = \frac{5}{8}\left[\frac{w}{hL^3}\right] x \left[L-x\right]\left[L^2 - Lx - a^2\right]}$$



c) If carries UDL throughout whole / entire span :-

$$\boxed{H = \frac{wL^2}{8h}}$$

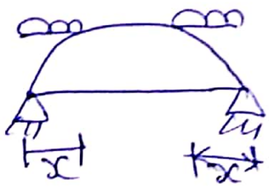


d) If UDL over half of the span

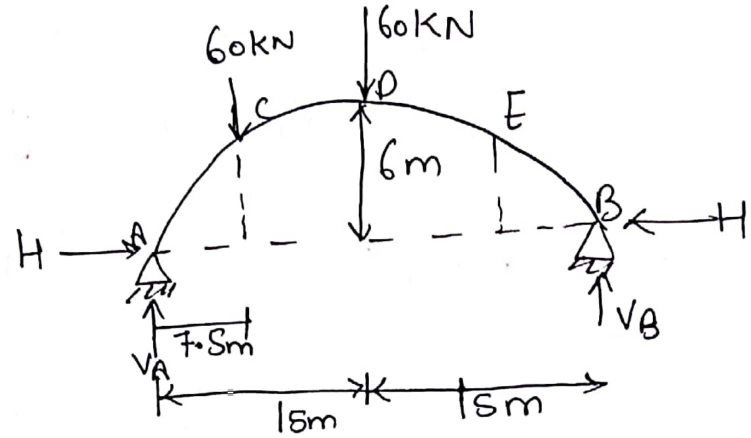
$$\boxed{H = \frac{wL^2}{16h}}$$

e) If UDL over a distance x from end.

$$\boxed{H = \left[\frac{w}{16hL^3}\right] x^2 \left(5L^3 - 5Lx^2 + 2x^3\right)}$$



Q→1 A two hinge parabolic arch of span 30m and rise 6m carries 2 point loads each of 60KN acting 7.5m & 15m from left end. Determine horizontal thrust and max +ve & -ve moment in the arch.



Sol
• Horizontal thrust [W = 60KN] x = 7.5m

$$H_1 = \frac{5}{8}\left[\frac{W}{hL^3}\right]x(L-x)(L^2+Lx-x^2)$$

$$H_1 = \frac{5}{8}\left[\frac{60}{6\times 30^3}\right]7.5(30-7.5)\left[30^2+30\times 7.5-7.5^2\right]$$

$$= 41.748 \text{ KN}$$

• 2nd Load = 60 KN   x = 15m

$$H_2 = \frac{25}{128}\left[\frac{WL}{h}\right] = \frac{25}{128}\left[\frac{60\times 30}{6}\right] = 58.594 \text{KN}$$

$$H = H_1 + H_2 = 41.748 + 58.594$$

$$\boxed{H = 100.342 \text{ KN}}$$

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b) $\Sigma M_A = 0$

$30 V_B = 60 \times 7.5 + 60 \times 15$

$V_B = \frac{60 \times 7.5 + 60 \times 15}{30} = 45 \text{ KN}$

b) $\Sigma f = 0$ $\quad V_A + V_B = 120$

$V_A = 120 - 45 = 75 \text{ KN}$

$y_c = \frac{4hx(L-x)}{L^2} = \frac{4 \times 6 \times 7.5(30-7.5)}{30^2}$

$\boxed{y_c = 4.5 \text{ m}}$

$M_c = V_A \times 7.5 - H y_c$

$= 75 \times 7.5 - 100.342 \times 4.5$

$M_c = 110.961 \text{ KNm}$

$M_D = V_B \times 15 - H \times h$

$= 45 \times 15 - 100.342 \times 6 = 72.948 \text{ KNm}$

$\boxed{\text{So Max +ve BM} = 110.961 \text{ KNm}}$

c) Max -ve BM occurs at RHS of arch. Let distance of -ve BM be z from right hand.

$M_E = V_B z - H y_z$

$Y_z = \frac{4hz(L-z)}{L^2} = \frac{4 \times 6 z(L-z)}{30^2}$

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$$M_E = 45z - 100.342\left[\frac{24}{30^2}\left[30z - z^2\right]\right]$$

$$= 45z - 2.676\left(30z - z^2\right]$$

$$\frac{d M_E}{dx} = 0$$

$$45 - 2.676(30 - 2z) = 0$$

$$45 = 80.28 - 5.352z$$

$$z = \frac{80.28 - 45}{5.352}$$

$$\boxed{z = 6.592}$$

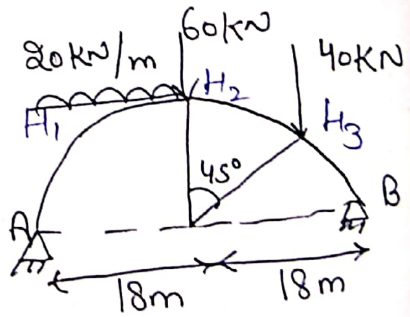
$$\boxed{y = 4.115}$$

$$M_E = 45 \times 6.592 - 100.342 \times 4.115$$

$$\boxed{M_E = -116.267 \text{ kNm}}$$

Q.2 Determine horizontal thrust develop in the semicircular arch of radius 18m.

Sol $H_1 = \frac{2}{3}\left[\frac{WR}{\pi}\right] = 76.394$



$$H_2 = \frac{W}{\pi} = 19.098$$

$$H_3 = \frac{W}{\pi}\cos^2\alpha = 6.366$$

$$H = H_1 + H_2 + H_3 = 101.858 \text{ kN}.$$

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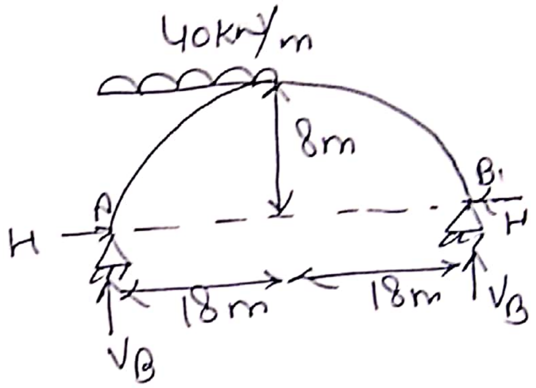
Q.3 A Parabolic arch having span 36m & rise 8m is loaded with an UDL of 40 kN/m at left half of arch.
Determine i) Horizontal thrust
ii) Max +ve & -ve moment
iii) Normal thrust & Radial Shear at 8m from left end.

Sol :



i) $H = \frac{wL^2}{16h} = \frac{40 \times 36^2}{16 \times 8}$

$H = 405 \, kN$

ii) $\Sigma M_A = 0$

$36 V_B = 40 \times 18 \times 9$

$V_B = \frac{40 \times 18 \times 9}{36} = 180 kN$

$\Sigma F = 0$

$V_A + V_B = 40 \times 18$

$V_A = (40 \times 18) - 180 = 540 kN$

iii) For AC :

$M_x = V_A x - \frac{40x^2}{2} - Hy_x$

$= 540x - 20x^2 - 405\left[\frac{4 \times 8}{36^2}(36x - x^2)\right]$

$= 540x - 20x^2 - 360x + 10x^2$

$= 180x - 10x^2$

$\frac{dM_x}{dx} = 180 - 20x = 0$

$20x = 180$

$\boxed{x = 9m}$

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$$M_x = 540\times9 - 20\times9^2 - 405\times\frac{32}{36^2}(36\times9 - 9^2)$$

$$\boxed{M_x = 810\ kNm}$$

iv) For CB :

$$M_x = V_B x - Hy_x$$

$$= 180x - 405\left[\frac{4\times8}{36^2}(36x - x^2)\right]$$

$$= 180x - 360x + 10x^2$$

$$= -180x + 10x^2$$

$$\frac{dM_x}{dx} = -180 + 20x$$

$$\boxed{x = 9m}$$

$$M_x = 180\times9 - 405\left[\frac{4\times8}{36^2}(36\times9 - 9^2)\right]$$

$$\boxed{M_x = -810\ kNm}$$

Max +ve moment = 810 kNm

Max −ve BM = −810 kNm

v) $V = V_A\times8 - 40\times8 = 540\times8 - 40\times8$

$$= 4000\ kN$$

$$\tan\theta = \frac{dy}{dx}$$

$$y = \frac{4hx}{L^2}[L - x] = \frac{4hx}{L} - \frac{4hx^2}{L}$$

$$\frac{dy}{dx} = \frac{4h}{L} - \frac{8hx}{L^2} = 0$$

$$\boxed{\frac{dy}{dx} = 0.493}$$

$$\boxed{\theta = \tan^{-1}0.493 = 26.24^\circ}$$

vi) Normal thrust :

$$N = V\sin\theta + H\cos\theta$$

$$N = 4000 \times \sin(26.24) + 405 \times \cos(26.24)$$

$$N = 1768.52 + 363.26$$

$$\boxed{N = 2131.78 \text{ kN}}$$

vii) Radial Shear

$$Q = V\cos\theta - H\sin\theta$$

$$= 4000 \times \cos(26.24) - 405 \times \sin(26.24)$$

$$3587.79 - 179.06$$

$$\boxed{Q = 3408.72 \text{ kN}}$$

- Effect of yielding / sinking of supports :

- If a structural support in a two hinge arch sinks by k or Δ then the expression for horizontal thrust is

$$\boxed{H = \frac{\int M'y \frac{dx}{EI}}{\int y^2 \frac{dx}{EI} + \Delta}}$$

Δ-amount of sinking

- Effect of sinking, rib shortening & temp :

- The horizontal thrust in a two hinge arch having all the above effects can be found out by :

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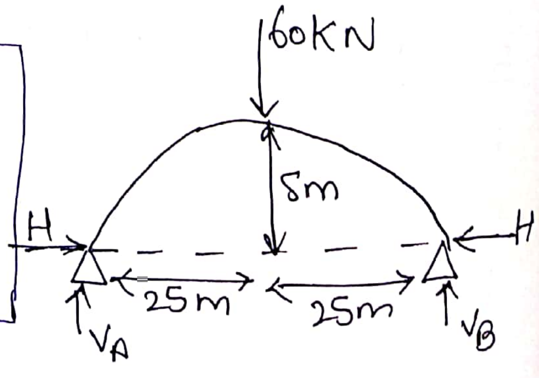
$$\boxed{H = \frac{\int M'y \frac{dx}{EI} + L\alpha t}{\int y^2 \frac{dx}{EI} + \frac{L}{EA_m} + \Delta}}$$

L = Length of arch
α = coff. of thermal expansion
t = temperature
Δ = amount of sinking
E = modulus of elasticity of steel
A_m = avg area of arch

Q→ A two hinge parabolic arch of span 50m and rise 5m is subjected to a central concentrated load of 60KN. It has an elastic support which yields by 0.0001 m/KN. Taking $E = 200\ KN/mm^2$, $I = 5\times10^9\ mm^4$. Avg area (A_m) = $1\times10^4\ mm^2$, $\alpha = 12\times10^{-6}/°C$. Calculate horizontal thrust developed when temperature rises by 20°C with considering rib shortening.

Sol :

$$\boxed{H = \frac{\int M'y \frac{dx}{EI} + L\alpha t}{\int y^2 \frac{dx}{EI} + \frac{L}{EA_m} + \Delta}}$$



$$y = \frac{4hx}{L^2}(L-x)$$

$$= \frac{4 \times 5 \times x}{50^2}[50-x]$$

a) $$\int M' y \frac{dx}{EI} = \frac{1}{EI}\int M' y \, dx = \frac{1}{EI}\left[2\int_0^{25} 30x\left(\frac{x}{125}\right)(50-x)dx\right]$$

$$= \frac{2}{EI}\left[\int_0^{25}(12x^2 - 0.24x^3)\,dx\right.$$

$$= \frac{2}{EI}\left[\frac{12 \times x^3}{3} - \frac{0.24x^4}{4}\right]_0^{25}$$

$$= \frac{2}{EI}\left[4 \times 25^3 - 0.06 \times 25^4\right]$$

$$= \frac{78125}{EI} = \frac{78125}{200 \times 10^6 \times 0.005} = \underline{0.390}$$

b) $$\int y^2 \frac{dx}{EI} = \frac{1}{EI}\int_0^{25} y^2 dx = \frac{2}{EI}\int_0^{25}\left(\frac{1}{125}(50-x)x\right)^2 dx$$

$$= \frac{2}{EI} \times \frac{1}{125^2}\left[\int_0^{25}(50^2 - 100x + x^2)x^2 dx\right]$$

$$= \frac{2}{EI} \times \frac{1}{125^2}\left[\frac{50^2 x^3}{3} - \frac{100x^4}{4} + \frac{x^5}{5}\right]_0^{25}$$

$$= \frac{2}{125^2 EI}\left[\frac{50^2 \times 25^3}{3} - \frac{100(25^4)}{4} + \frac{(25)^5}{5}\right]$$

$$= \frac{666.67}{EI} = \underline{0.00067}$$

$$H = \cancel{78125}\ \frac{0.390 + L\alpha t}{0.00067 + \frac{L}{EA_m} + \Delta} = \frac{0.390 + 0.012}{0.00067 + 0.000025 + 0.0001}$$

$$H = \frac{0.402}{7.2 \times 10^{-4}} = \underline{558.33\ \text{KN}}$$